ME 1065 - LMTD and Effectiveness-NTU Derivations

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Brief Derivation of the LMTD

To design or predict the performance of a heat exchanger, the **LMTD** and the **effectiveness-NTU** methods are both useful. I'll first touch on the LMTD method, to give you an overview of its derivation and meaning.

One circumstance in designing or predicting the performance of a hxgr is the need to relate the heat transfer rate to quantities like the inlet and outlet temperatures, the U and the A.

Examine an exchange of heat between two streams, separated by a thin sheet of area A, as seen in the next Figure. Let's write the expressions for mass conservation and energy conservation, and see what assumptions we need to make. Also need to determine the control volume.

Assume:

- Uniform
- Steady flow
- All the heat that comes from the hot stream goes into the cold stream (the hxgr is insulated from its surroundings)
- No phase change
- Constant specific heats
- Negligible Kinetic and potential energy
- U is constant (or nearly so)

Mass conservation:

$$\frac{\partial}{\partial t}\int_{CV}\rho d\forall + \int_{CS}\rho\vec{V}\cdot\hat{n}dA = 0$$

$$\int_{CS}\rho\vec{V}\cdot\hat{n}dA = 0$$

$$\dot{m}_c = \rho_{c,o}V_oA_o = \rho_{c,i}V_iA_i$$

Energy conservation:

$$\frac{\partial}{\partial t}\int_{CV}\rho e d\forall + \int_{CS}\rho e\vec{V}\cdot\hat{n}dA = \dot{Q} - \dot{W}^*$$

$$\int_{CS}\rho(e+Pv)\vec{V}\cdot\hat{n}dA = 0$$

$$\dot{Q} = \rho_{c,o}h_{c,o}V_oA_o - \rho_{c,i}h_{c,i}V_iA_i$$

$$\dot{Q} = \dot{m}_c(h_{c,o} - h_{c,i})$$

$$\dot{Q} = \dot{m}_h(h_{h,i} - h_{h,o})$$

So with the specific heat assumption that we made (and treating the fluids as incompressible), we can write

$$\dot{Q} = \dot{m}c_p\Delta T$$

We'll also define $C_i = \dot{m}_i c_{p,i}$ as the heat capacity rate. We want to relate the inlet and outlet temperatures, the U, and the A, to the rate of heat transferred, and we can do this as:

$$\dot{Q} = UA\Delta T_{mean}$$

.

We now need to find what this mean temperature is. To do this, we'll look at a differential area of the heat exchanger where a differential amount of heat is transferred.

Using this equation to examine the hot and cold streams separately yields:

$$\delta\dot{Q} = -C_h\ dT_h\ ;\ C_h = (\dot{m}\ c_p)_h \tag{1}$$

$$\delta\dot{Q} = \pm C_c\ dT_c\ ;\ C_c = (\dot{m}\ c_p)_c \tag{2}$$

where $\pm$ or $\mp$ appears, the top sign designates parallel-flow, and the bottom sign designates counterflow.

A heat exchanger relationship for expressing the heat transfer between fluids over a differential area is:

$$\delta\dot{Q} = U\ (T_h - T_c)\ dA\ \text{, or} \tag{3}$$

$$T_h - T_c = \frac{\delta\dot{Q}}{U\ dA} \tag{4}$$

Rearranging equations 1 and 2 yields:

$$dT_h = -\frac{\delta\dot{Q}}{C_h}\ \text{and}\ dT_c = \pm\frac{\delta\dot{Q}}{C_c}\ \text{; so}$$

$$dT_h - dT_c = d(T_h - T_c) = \delta\dot{Q}\ (-\frac{1}{C_h} \mp \frac{1}{C_c}) \tag{5}$$

Dividing the above equation by equation 4 results in:

$$\frac{d(T_h - T_c)}{T_h - T_c} = U\ (-\frac{1}{C_h} \mp \frac{1}{C_c})\ dA \tag{6}$$

Equation 6 can then be integrated over the heat exchanger:

Parallel-flow:

$$ln[\frac{T_{h_2} - T_{c_2}}{T_{h_1} - T_{c_1}}] = UA\ (-\frac{1}{C_h} - \frac{1}{C_c})$$

Counterflow:

$$ln[\frac{T_{h_2} - T_{c_1}}{T_{h_1} - T_{c_2}}] = UA\ (-\frac{1}{C_h} + \frac{1}{C_c}) \tag{7}$$

Replacing the specific heats of equation 7 with those found before yields

Parallel-flow:

$$ln[\frac{T_{h_2} - T_{c_2}}{T_{h_1} - T_{c_1}}] = \frac{UA}{\dot{Q}}[(T_{h_2} - T_{h_1}) + (T_{c_1} - T_{c_2})]$$

Counterflow:

$$ln[\frac{T_{h_2} - T_{c_1}}{T_{h_1} - T_{c_2}}] = \frac{UA}{\dot{Q}}[(T_{h_2} - T_{h_1}) + (T_{c_2} - T_{c_1})]$$

$$\dot{Q} = UA\ \frac{\Delta T_1 - \Delta T_2}{ln[\frac{\Delta T_1}{\Delta T_2}]}$$

$$\Delta T_{LogMean} = \frac{\Delta T_1 - \Delta T_2}{ln[\frac{\Delta T_1}{\Delta T_2}]} \quad (8)$$

This equation is valid only if $C_c \neq C_h$; otherwise, $\Delta T_1 = \Delta T_2$ and a denominator of zero results.

Example 11.1 from Incrop. and DeW. is a good example that incorporates the convection coefficient and the LMTD.

Effectiveness-NTU method

If only the inlet temperatures (and not the outlet temperatures) are known, the LMTD method requires iteration. In these cases (and in some others), the effectiveness-NTU method should be used instead ($\varepsilon - NTU$). **REMEMBER: $\dot{Q} = q$.**

We want to show that $\varepsilon = \varepsilon(NTU, \frac{C_{min}}{C_{max}})$ We'll first show that this is true for parallel-flow, where $C_h = C_{min}$.

So, what is **q** equal to, in terms of the temperatures and heat capacity rate of the hot fluid?

$$q = C_h(T_{hi} - T_{ho})$$

$$\varepsilon = \frac{C_h(T_{hi} - T_{ho})}{C_{min}(T_{hi} - T_{ci})} = \frac{T_{hi} - T_{ho}}{T_{hi} - T_{ci}}$$

$$C_r = \frac{C_{min}}{C_{max}} = \frac{C_h}{C_c} = \frac{T_{co} - T_{ci}}{T_{hi} - T_{ho}}$$

Now, returning to the derivation of the LMTD, we know that for a parallel-flow hxgr,

$$\ln(\frac{\Delta T_2}{\Delta T_1}) = -UA(\frac{1}{C_h} + \frac{1}{C_c}) = -\frac{UA}{C_h}(1 + \frac{C_h}{C_c}) = -\frac{UA}{C_{min}}(1 + C_r)$$

$$\ln(\frac{T_{ho} - T_{co}}{T_{hi} - T_{ci}}) = -NTU(1 + C_r)$$

and

$$\frac{T_{ho} - T_{co}}{T_{hi} - T_{ci}} = \exp(-NTU(1 + C_r))$$

Now we have three expressions for temperature that relate to ε, NTU, and C_r, and we can eliminate temperature altogether to find ε only in terms of NTU and C_r. The derivation is as follows:

$$T_{co} = C_r(T_{hi} - T_{ho}) + T_{ci}$$

$$\frac{T_{ho} - T_{co}}{T_{hi} - T_{ci}} = \frac{(T_{ho} - T_{hi}) + (T_{hi} - T_{co})}{T_{hi} - T_{ci}}$$

$$\frac{T_{ho} - T_{co}}{T_{hi} - T_{ci}} = \frac{T_{ho} - T_{hi}}{T_{hi} - T_{ci}} + \frac{T_{hi} - T_{ci}}{T_{hi} - T_{ci}} - C_r\frac{T_{hi} - T_{ho}}{T_{hi} - T_{ci}}$$

$$\frac{T_{ho} - T_{co}}{T_{hi} - T_{ci}} = (-\varepsilon) + (1) - C_r(\varepsilon) = \exp(-NTU(1 + C_r))$$

and

$$\varepsilon = \frac{1 - \exp(-NTU(1 + C_r))}{1 + C_r}$$

We could also go through and do the same derivation for $C_c = C_{min}$. You'll find more expressions for ε and NTU in any heat transfer textbook.